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On the stability of fine-scaled turbulent free shear flows

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A theoretical model has been developed to investigate the stability of a disturbance in an incompressible turbulent shear flow dominated by turbulence scales that are small with respect to the cross-stream dimension of the flow. The approach utilizes the 'phase average' concept to derive the differential equations governing the mechanics of a potential flow disturbance. Turbulence closure is effected at second order. The result is an Orr–Sommerfeld-type equation with complications introduced by the turbulence model. Integration of the linear eigenvalue problem for a wake disturbance leads to the result that the critical eddy-viscosity-based Reynolds number is markedly increased by decreasing the turbulence scale. The viscoelastic behaviour of background turbulence, further complicated by the effects of mean shear, appears to provide stabilization and is discussed in some detail.

1. Introduction

Everyone has undoubtedly observed the calm wake behind a large steamship as it traverses the open sea. The near wake is rather homogeneous in appearance and dominated by turbulence from the ship's boundary layer. A particularly intriguing qualitative aspect of such a wake is its apparent ability to damp surface waves incident upon its turbulent structure. Something similar seems to occur in the wake of a hypersonic vehicle surrounded by a turbulent boundary layer as first described by Finson (1973). He observed that the wake diffused very slowly and appeared to consist of rather fine-scaled turbulence that originated in the boundary layer. A second-order turbulence closure model was used to compute the observed wake development. In addition, he recognized that the wake must be stabilized to large-scale laminar-type instabilities by the fine-scaled turbulence, since such instabilities would rapidly generate large-scale, intense turbulence that would be inconsistent with the observed low diffusion rate.

The purpose of this paper is to investigate the stability of a free shear flow containing fine-scaled background turbulence. By fine-scaled it is meant that the length scale of the turbulence should be small compared to the wavelength of potentially unstable modes. For computational purposes, we limit ourselves to the specific case of a two-dimensional incompressible wake. A more general objective will be to elucidate the nature of the interaction between background turbulence and a potential wave disturbance. In an interesting paper, Crow (1968) showed that the background turbulence generally behaves in a viscoelastic manner. The elastic limit pertains when

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the relaxation rate of the turbulence is much smaller than the imposed rate of deformation, and Crow was able to derive the effective elastic modulus for isotropic turbulence from purely kinematic considerations. He also obtained an approximate solution for the effective eddy viscosity in the viscous limit (fast turbulence relaxation).

Hussain & Reynolds (1970*a*, *b*, 1972), and Reynolds & Hussain (1972) provided the formalism required to study the behaviour of a perturbation in a turbulent shear flow. They introduced the concept of the 'phase average' or 'periodic average' which was used to develop equations for a set of wave-perturbed quantities, in addition to the usual time-averaged variables. An analogous averaging technique has been employed by investigators studying the effect of a turbulent wind on ocean surface waves (e.g. Phillips 1966). Mollo-Christensen (1971) distinguished the two averaging processes by considering the turbulence to be characterized by two substantially different length scales. Hussain & Reynolds showed that the background turbulence affects a wave disturbance through the wave-induced oscillation in the Reynolds stress. They used an eddy-viscosity closure approximation to compute this induced Reynolds stress, and obtained reasonable agreement with the results of their experiment in a turbulent channel flow. However, they recommended a second-order closure of the type developed here as the obvious improvement.

Reynolds (1972) employed a similar eddy-viscosity closure approximation to obtain a stability map for large-scale disturbances in an incompressible two-dimensional turbulent wake. In so doing it was necessary to consider carefully the jump conditions at the outer edge of the turbulent zone (superlayer), where the viscosity is discontinuous. However, the neutral stability characteristics obtained by Reynolds are not essentially different from those in a laminar wake, with the molecular viscosity replaced by the eddy viscosity. This result does not explain the stabilizing influence of fine-scaled background turbulence discussed at the outset. Assuming the eddy viscosity to be proportional to the scale size, the fine-scale-dominated wake would have a larger effective Reynolds number, and hence be more unstable, than would a standard turbulent wake, according to Reynolds' (1972) result.

The more general viscoelastic behaviour of fine-grained turbulence would suggest more complicated stability characteristics. For example, Betchov (1966) obtained a shift of the Blasius boundary-layer neutral stability curve by using a complex viscosity to model the visco-elastic properties of polymer additives. The approach taken in the present study is to effect closure at second order. It will be shown how one naturally obtains the expected results in the viscous and elastic limits, and we have elaborated upon the work of Crow (1968) by adding the features of the mean shear.

The approach taken here is rather similar to that of Elswick (1971),† although for different purposes. In the present paper, we determine the conditions leading to the 'apparent' stability of fine-scaled turbulent shear flows. Elswick, on the other hand, sought to establish whether the large eddies observed in turbulent shear flows can be attributed to instability of the basic flow. We specifically consider the characteristics of a shear flow (wake) that is easily stabilized whereas Elswick considered the characteristics of the shear layer, which is perhaps most unstable. Both studies recognize, however, that some kind of viscoelastic response of perturbations of the background turbulence should be anticipated in a proper treatment of the turbulent flow stability

† The authors are indebted to a reviewer for bringing this unpublished thesis to their attention.

problem. Elswick employed a somewhat less detailed second-order closure scheme than that utilized here. He obtained a lower branch of the neutral stability curve for the turbulent shear layer, and found a critical obliquity (about 50°) of the disturbance to the main stream above which his assumption of homogeneous background turbulence apparently becomes inappropriate.

2. Basic equations

The specific problem to be addressed is that of the stability of an incompressible, two-dimensional wake containing background turbulence. It will be tacitly assumed that there is a wide separation between the energy-containing wavenumbers of the turbulence and potentially unstable wavenumbers, i.e. that the background turbulence is relatively fine-scaled. Sufficiently large-scaled turbulence would merely serve as initial disturbances for the standard laminar-type instability process. We further presume that the required characteristics of the mean flow (mean velocity) and turbulence (intensity, scale size, etc.) may be specified.

In order to study the development of a wave disturbance, we adopt the concept of a periodic or phase average. Several authors, most notably Hussain & Reynolds (1970a,b, 1972) and Reynolds & Hussain (1972), have used this technique. It involves decomposing any flow property such as the velocity $u(\mathbf{x},t)$ into a mean value $\bar{u}(\mathbf{x})$, a contribution $\tilde{u}(\mathbf{x},t)$, due to the wave, and the contribution $u'(\mathbf{x},t)$ from the background turbulence:

$$u(\mathbf{x},t) = \bar{u}(\mathbf{x}) + \tilde{u}(\mathbf{x},t) + u'(\mathbf{x},t). \tag{1}$$

The conventional time average for any flow variable f is defined as

$$f(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\mathbf{x},t)\,dt \tag{2}$$

and the periodic average is defined as

$$\langle f(\mathbf{x},t)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} f(\mathbf{x},t+n\tau), \tag{3}$$

where τ is the period of the wave. The periodic average may be considered to be an ensemble average at a fixed value of the phase of the wave or, for a two-dimensional wave, to be an average over the third dimension. The properties of the periodic average are described in detail by Hussain & Reynolds (1970a) and by Reynolds & Hussain (1972). We note in particular that

$$\langle f\rangle = \bar{f} + \tilde{f}. \tag{4}$$

It is considered that the background turbulence is random and not correlated with the wave, so that $\langle f'\rangle = 0$.

The governing equations for the wave disturbance $\tilde{u}_i$ are obtained by applying the periodic and time averages to the continuity and momentum equations. It is easily shown that each velocity component satisfies the usual incompressible continuity relation

$$\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial u_i'}{\partial x_i} = 0. \tag{5}$$

We shall not write down the momentum equation for the mean velocity or the fluctuating turbulent component. As long as the wave disturbance strength is infinitesimal, the mean and turbulent fields are unaffected by the wave. The momentum equation for the wave disturbance is

$$\frac{\partial \tilde{u}_i}{\partial t} + \bar{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} - \tilde{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{\tilde{u}_i \tilde{u}_j} - \tilde{u}_i \tilde{u}_j) - \frac{\partial}{\partial x_j}(\langle u_i' u_j' \rangle - \overline{u_i' u_j'}). \quad (6)$$

It is apparent that the final term in this equation contains the influence of the background turbulence on the wave disturbance. The preceding terms are those which would be obtained in a standard viscous-stability problem. The final term involves the difference between the periodic and time averages of the turbulent Reynolds stress. Thus the wave experiences a stress which is the oscillation induced in the Reynolds stress by the wave. Following Reynolds & Hussain (1972) we introduce $\tilde{r}_{ij}$ for the negative of this stress

$$\tilde{r}_{ij} = \langle u_i' u_j' \rangle - \overline{u_i' u_j'}. \quad (7)$$

The term containing $\tilde{r}_{ij}$ in (6) is analogous to the familiar Reynolds stress term in the mean momentum equation for turbulent flows. Its appearance introduces a closure problem. One approach (used by Reynolds & Hussain 1972 to analyse their channel flow data and by Reynolds 1972 for wake stability calculations) is to introduce a gradient diffusion approximation for $\tilde{r}_{ij}$. The obvious choice for the diffusivity is the eddy viscosity of the background turbulence:

$$\tilde{r}_{ij} = \epsilon\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right). \quad (8)$$

With this closure approximation, the wake stability problem essentially reduces to the laminar stability problem, except perhaps for some difference in the radial profile of the viscosity. Our approach will be to attempt the derivation of a more basic equation for $\tilde{r}_{ij}$, and we do this by invoking closure at second order.

The formally exact governing equation for $\tilde{r}_{ij}$ has been given by Reynolds & Hussain (1972):

$$\begin{aligned}\frac{\partial \tilde{r}_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tilde{r}_{ij}}{\partial x_k} = & -\left(\tilde{r}_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tilde{r}_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) - \tilde{u}_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} \\ & - \left(\overline{u_j' u_k'}\frac{\partial \tilde{u}_i}{\partial x_k} + \overline{u_i' u_k'}\frac{\partial \tilde{u}_j}{\partial x_k}\right) + \overline{\tilde{u}_k \frac{\partial \tilde{r}_{ij}}{\partial x_k}} - \tilde{u}_k \frac{\partial \tilde{r}_{ij}}{\partial x_k} \\ & + \overline{\tilde{r}_{jk}\frac{\partial \tilde{u}_i}{\partial x_k}} - \tilde{r}_{jk}\frac{\partial \tilde{u}_i}{\partial x_k} + \overline{\tilde{r}_{ik}\frac{\partial \tilde{u}_j}{\partial x_k}} - \tilde{r}_{ik}\frac{\partial \tilde{u}_j}{\partial x_k} \\ & + \frac{\partial}{\partial x_k}(\overline{u_i' u_j' u_k'} - \langle u_i' u_j' u_k' \rangle) + \frac{\partial}{\partial x_i}(\overline{p' u_j'} - \langle p' u_j' \rangle) \\ & + \frac{\partial}{\partial x_j}(\overline{p' u_i'} - \langle p' u_i' \rangle) - \overline{p' \frac{\partial u_j'}{\partial x_i}} + \left\langle p' \frac{\partial u_j'}{\partial x_i}\right\rangle - \overline{p' \frac{\partial u_i'}{\partial x_j}} \\ & + \left\langle p' \frac{\partial u_i'}{\partial x_j}\right\rangle + \nu \frac{\partial^2 \tilde{r}_{ij}}{\partial x_k \partial x_k} + 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - 2\nu\left\langle \frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\right\rangle. \end{aligned} \quad (9)$$

This equation is not quite as forbidding as it first appears. The left-hand side obviously represents convection. The first three types of terms on the right may be called production terms, and require no closure. The next six terms are second order in wave amplitude and will be omitted for present purposes. The terms involving triple correlations represent turbulent diffusion of $\tilde{r}_{ij}$. Then we have terms involving pressure fluctuations, which tend to drive the turbulence toward isotropy. The viscous diffusion term is, of course, small at high Reynolds numbers. Finally there are dissipation terms.

Equation (9) will be useful only if a closure scheme can be developed for the turbulent diffusion (triple correlation), pressure fluctuation, and dissipation terms. This cannot, of course, be done in a rigorous manner. However, reasonably useful approximations have been developed for turbulent flow computations (i.e. for the time averages, in situations without an imposed wave disturbance), and it will be assumed here that similar approximations can be applied to periodic-averaged quantities as well. The particular expressions to be used are taken from Rotta (1951) and Hanjalić & Launder (1972).

Let us first consider the dissipation term. At sufficiently high Reynolds numbers the dissipation rate should be isotropic. The common closure for the time-averaged rate is

$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = -\frac{2k_d}{3}\frac{q}{\Lambda}q^2\delta_{ij}. \qquad (10a)$$

Here q^2 is the turbulent kinetic energy $\frac{1}{2}\overline{u_{kk}'^2}$ and Λ the macroscale of the relaxation rate. Finson (1973) found that grid turbulence data indicates 0·4 to be the appropriate value for the constant k_d. Considering this rate to be unaffected by the wave disturbance, the analogy to (10a) for the periodic average is

$$2\nu\left\langle\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\right\rangle = -\frac{2k_d}{3}\frac{q}{\Lambda}\frac{\langle u_{kk}'^2\rangle}{2}\delta_{ij}. \qquad (10b)$$

The difference between (10a) and (10b) then yields

$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - 2\nu\left\langle\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\right\rangle = -\frac{2k_d}{3}\frac{q}{\Lambda}\frac{\tilde{r}_{kk}}{2}\delta_{ij}. \qquad (10c)$$

The manner in which pressure fluctuations drive turbulence toward isotropy has been modelled by Rotta (1951):

$$\overline{p'\frac{\partial u_i'}{\partial x_j}} + \overline{p'\frac{\partial u_j'}{\partial x_i}} = -k_p\frac{q}{\Lambda}(\overline{u_i'u_j'} - \tfrac{2}{3}q^2\delta_{ij}) + (\bar{a}_{lj}^{mi} + \bar{a}_{li}^{mj})\frac{\partial\bar{u}_l}{\partial x_m}. \qquad (11a)$$

Data on the return of grid turbulence to isotropy indicates a value of 0·48 for k_p (Finson 1973). The need for the term involving the mean shear was recognized by Rotta (1951). We adopted a somewhat simplified version of the expression suggested by Hanjalic & Launder (1972) for $\bar{a}_{lj}^{mi}$:

$$\bar{a}_{lj}^{mi} = \tfrac{8}{15}\delta_{mi}\delta_{lj}q^2 - \tfrac{2}{15}(\delta_{ml}\delta_{ij} + \delta_{mj}\delta_{il})q^2 + \tfrac{10}{11}\delta_{lj}s_{mi} - \tfrac{2}{11}(\delta_{ij}s_{ml} + \delta_{il}s_{mj} + \delta_{ml}s_{ij} + \delta_{mj}s_{il}), \qquad (11b)$$

where $$s_{ij} = \overline{u_i'u_j'} - \tfrac{2}{3}q^2\delta_{ij}.$$

The obvious requirements of symmetry with respect to i, j and continuity ($\bar{a}_{lk}^{mk} = 0$) are satisfied with this form. In fact, the use of these requirements to determine the fractional coefficients in (11*b*) guarantees that the proper stresses result in the elastic limit, to be discussed below.†

By analogy with (11*a*), it may be argued that the appropriate form for the periodic-averaged pressure fluctuation terms is

$$\left\langle p'\frac{\partial u_i'}{\partial x_j}\right\rangle+\left\langle p'\frac{\partial u_j'}{\partial x_i}\right\rangle=-k_p\frac{q}{\Lambda}(\langle u_i'u_j'\rangle-\tfrac{1}{3}\langle u_k'u_k'\rangle\delta_{ij})+(\tilde{a}_{lj}^{mi}+\tilde{a}_{li}^{mj})\frac{\partial\bar{u}_l}{\partial x_m}+(\bar{a}_{lj}^{mi}+\bar{a}_{li}^{mj})\frac{\partial\tilde{u}_l}{\partial x_m},\quad(11c)$$

where $\tilde{a}_{lj}^{mi}$ denotes (11*b*) with time averages replaced by periodic averages ($\frac{1}{2}\langle u_k'u_k'\rangle$ instead of q^2). If we then subtract (11*c*) from (11*a*) we obtain the desired closure approximation for the pressure fluctuation terms

$$-\overline{p'\frac{\partial u_i'}{\partial x_j}}+\left\langle p'\frac{\partial u_i'}{\partial x_j}\right\rangle-\overline{p'\frac{\partial u_j'}{\partial x_i}}+\left\langle p'\frac{\partial u_j'}{\partial x_i}\right\rangle=-k_p\frac{q}{\Lambda}(\tilde{r}_{ij}-\tfrac{1}{3}\tilde{r}_{kk}\delta_{ij})+(\tilde{a}_{lj}^{mi}+\tilde{a}_{li}^{mj})\frac{\partial\bar{u}_l}{\partial x_m}+(\bar{a}_{lj}^{mi}+\bar{a}_{li}^{mj})\frac{\partial\tilde{u}_l}{\partial x_m}.\quad(11d)$$

Finally there are the turbulent diffusion and pressure diffusion terms, which we shall neglect. This is admissible if the eddy viscosity of the background turbulence is not too large, which will generally be the case for fine-scaled background turbulence. If d_w is the wake width, the diffusion terms should be of order $\epsilon\tilde{r}_{ij}/d_w^2$. For comparison the production terms $\tilde{r}_{ik}\,\partial\bar{u}_j/\partial x_k$ should be $\sim\tilde{r}_{ij}\Delta\bar{u}/d_w$. Thus

$$\frac{\text{diffusion}}{\text{production}}\sim\frac{\epsilon}{\Delta\bar{u}d_w}.\quad(12)$$

This ratio is less than unity even for equilibrium wake turbulence (Townsend 1956). In the wake of a slender body where the background turbulence is residual boundary-layer turbulence, as discussed by Finson (1973), background turbulence should be relatively weak and neglect of the diffusion terms would be quite well justified.

Inserting (10*c*) and (11*d*) into (9), and omitting the several terms that have been argued to be negligible, yields the governing relation for the wave stress $\tilde{r}_{ij}$:

$$\frac{D\tilde{r}_{ij}}{Dt}=-\left(\tilde{r}_{ik}\frac{\partial\bar{u}_j}{\partial x_k}+\tilde{r}_{jk}\frac{\partial\bar{u}_i}{\partial x_k}\right)-\tilde{u}_k\frac{\partial\overline{u_i'u_j'}}{\partial x_k}-\left(\overline{u_j'u_k'}\frac{\partial\tilde{u}_i}{\partial x_k}+\overline{u_i'u_k'}\frac{\partial\tilde{u}_j}{\partial x_k}\right)-k_p\frac{q}{\Lambda}(\tilde{r}_{ij}-\tfrac{1}{3}\tilde{r}_{kk}\delta_{ij})+(\tilde{a}_{lj}^{mi}+\tilde{a}_{li}^{mj})\frac{\partial\bar{u}_l}{\partial x_m}+(\bar{a}_{lj}^{mi}+\bar{a}_{li}^{mj})\frac{\partial\tilde{u}_l}{\partial x_m}-\tfrac{1}{3}k_d\frac{q}{\Lambda}\tilde{r}_{kk}\delta_{ij}.\quad(13)$$

† Subsequent to the effort described here, Launder, Reece & Rodi (1975) and the authors (unpublished) have found that the inclusion of an additional term in (11*b*) involving s_{ij} yields more accurate agreement with data from experiments such as that of Champagne, Harris & Corrsin (1970). However, such a term is unimportant in either the viscous or elastic limits and its omission is not expected to have a significant effect on the results presented here.

This is still a relatively complicated equation, particularly in view of the fact that it couples different components of $\tilde{r}_{ij}$. The selection of a minimum self-consistent set of $\tilde{r}_{ij}$'s will be discussed in the following section, where the eigenvalue problem is formulated.

It must be admitted that the present application of second-order closure is rather bold. There is little direct evidence, even from such a sophisticated experiment as that of Hussain & Reynolds (1972), to offer support. Intuitively, it may be argued that the above closure approximations should be permissible (or at least as permissible as in 'standard' turbulent shear flows) if the wave frequency $2\pi/\tau$ is small compared to the turbulent relaxation rate q/Λ. In the opposite limit of very high wave frequency ($2\pi/\tau \gg q/\Lambda$), the relaxation terms requiring closure prove to be negligible in comparison to the inertial terms. Thus the above equation for $\tilde{r}_{ij}$ should be accurate in the opposite limits of zero and infinite wave frequency, and we can only hope that it will be reasonable at intermediate frequencies. In the final section we shall elaborate on the behaviour of the wave stress equation in various limits, and it will be seen that the equation possesses several interesting and not unexpected properties.

3. Two-dimensional wake stability calculations

In this section we describe the simultaneous solution of (6) for the wave perturbation velocity and (13) for the wave stress, as a linear stability problem. In essence, it is possible to substitute (13) into (6) to obtain a fourth-order Orr–Summerfeld type of equation with complicated viscous terms.

We consider a two-dimensional wake flow with a mean velocity, $\overline{u}_1(x_2)$. Subscripts 1 and 2 refer to directions along the main flow and directions normal to the main flow, respectively. Since we consider the amplitudes of the wave field properties to be small, quadratic terms in wave quantities (e.g., $\tilde{u}_i \tilde{u}_j$) are neglected. The wave stability equations then read:

$$\frac{\partial \tilde{u}_1}{\partial t} + \overline{u}_1 \frac{\partial \tilde{u}_1}{\partial x_1} + \tilde{u}_2 \frac{\partial \overline{u}_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_1} - \left(\frac{\partial \tilde{r}_{11}}{\partial x_1} + \frac{\partial \tilde{r}_{12}}{\partial x_2}\right), \tag{14}$$

$$\frac{\partial \tilde{u}_2}{\partial t} + \overline{u}_1 \frac{\partial \tilde{u}_2}{\partial x_1} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_2} - \left(\frac{\partial \tilde{r}_{21}}{\partial x_1} + \frac{\partial \tilde{r}_{22}}{\partial x_2}\right). \tag{15}$$

For the two-dimensional wake considered here, there are four independent, non-zero components of the wave-induced Reynolds stress tensor: $\tilde{r}_{11}, \tilde{r}_{22}, \tilde{r}_{33}$, and $\tilde{r}_{12} = \tilde{r}_{21}$. Using the boundary-layer approximation to retain only cross-stream derivatives of the mean and turbulent properties, the equations for these four stress components are:

$$\begin{aligned}\frac{\partial \tilde{r}_{11}}{\partial t} + \overline{u}_1 \frac{\partial \tilde{r}_{11}}{\partial x_1} = {} & -\frac{2}{3}\tilde{u}_2 \frac{\partial q^2}{\partial x_2} - \frac{10}{11}\tilde{r}_{12}\frac{\partial \overline{u}_1}{\partial x_2} - \frac{8}{15}q^2 \frac{\partial \tilde{u}_1}{\partial x_1} \\ & - \frac{8}{11}\overline{u_1' u_2'}\left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1}\right) - \frac{2}{11}\overline{u_1' u_2'}\frac{\partial \tilde{u}_1}{\partial x_2} + \tilde{r}_{11}\frac{q}{\Lambda}\left(-\frac{2k_p}{3} - \frac{k_d}{2}\right) \\ & + \tilde{r}_{22}\frac{q}{\Lambda}\left(\frac{k_p}{3} - \frac{k_d}{2}\right) + \tilde{r}_{33}\frac{q}{\Lambda}\left(\frac{k_p}{3} - \frac{k_d}{2}\right),\end{aligned} \tag{16}$$

$$\frac{\partial \tilde{r}_{12}}{\partial t}+\bar{u}_1\frac{\partial \tilde{r}_{12}}{\partial x_1} = -\tilde{u}_2\frac{\partial \overline{u_1' u_2'}}{\partial x_2}-\frac{4}{15}q^2\left(\frac{\partial \tilde{u}_1}{\partial x_1}+\frac{\partial \tilde{u}_2}{\partial x_1}\right)-k_p\frac{q}{\Lambda}\tilde{r}_{12}$$
$$+\tilde{r}_{11}\frac{\partial \bar{u}_1}{\partial x_2}\left(-\frac{9}{55}\right)+\tilde{r}_{22}\frac{\partial \bar{u}_1}{\partial x_2}\left(-\frac{14}{55}\right)+\tilde{r}_{33}\frac{\partial \bar{u}_1}{\partial x_2}\left(\frac{1}{55}\right) \quad (17)$$

$$\frac{\partial \tilde{r}_{22}}{\partial t}+\bar{u}_1\frac{\partial \tilde{r}_{22}}{\partial x_1} = -\frac{2}{3}\tilde{u}_2\frac{\partial q^2}{\partial x_2}-\frac{8}{11}\tilde{r}_{12}\frac{\partial \bar{u}_1}{\partial x_2}+\frac{8}{15}q^2\frac{\partial \tilde{u}_1}{\partial x_1}$$
$$-\frac{8}{11}\overline{u_1' u_1'}\left(\frac{\partial \tilde{u}_1}{\partial x_2}+\frac{\partial \tilde{u}_2}{\partial x_1}\right)-\frac{2}{11}\overline{u_1' u_2'}\frac{\partial \tilde{u}_2}{\partial x_1}+\tilde{r}_{11}\frac{q}{\Lambda}\left(\tfrac{1}{3}k_p-\tfrac{1}{2}k_d\right)$$
$$+\tilde{r}_{22}\frac{q}{\Lambda}\left(-\frac{2k_p}{3}-\frac{k_d}{2}\right)+\tilde{r}_{33}\frac{q}{\Lambda}\left(\frac{k_p}{3}-\frac{k_d}{2}\right), \quad (18)$$

$$\frac{\partial \tilde{r}_{33}}{\partial t}+\bar{u}_1\frac{\partial \tilde{r}_{33}}{\partial x_1} = -\frac{2}{3}\tilde{u}_2\frac{\partial q^2}{\partial x_2}-\frac{4}{11}\tilde{r}_{12}\frac{\partial \bar{u}_1}{\partial x_2}-\frac{4}{11}\overline{u_1' u_2'}\left(\frac{\partial \tilde{u}_2}{\partial x_1}+\frac{\partial \tilde{u}_1}{\partial x_2}\right)$$
$$+\tilde{r}_{11}\frac{q}{\Lambda}\left(\frac{k_p}{3}-\frac{k_d}{2}\right)+\tilde{r}_{22}\frac{q}{\Lambda}\left(\frac{k_p}{3}-\frac{k_d}{2}\right)+\tilde{r}_{33}\frac{q}{\Lambda}\left(-\frac{2k_p}{3}-\frac{k_d}{2}\right). \quad (19)$$

The complexity of these equations can be reduced somewhat by noting that the $\tilde{r}_{33}$ component barely enters the two-dimensional stability problem. It does not enter (14) or (15) for the perturbation velocities, and appears only as the final term in each of (16)–(18). With the values quoted above, $k_d = 0{\cdot}40$ and $k_p = 0{\cdot}48$, it is readily seen that the coefficient of the $\tilde{r}_{33}$ term is an order of magnitude smaller than the leading relaxation term in each equation (that containing $\tilde{r}_{11}$ in (16), etc.). Henceforth, we will use the reduced system of (16)–(18) without the $\tilde{r}_{33}$ terms. This does not imply that $\tilde{r}_{33} = 0$; rather, it means that $\tilde{r}_{33}$ is not significant to the solution of (14)–(18).

We may solve the linear system of (14)–(18), by assuming the wake to be a parallel flow in the x_1 direction; we can then Fourier decompose the unknowns in the following manner:

$$[\tilde{u}_1, \tilde{u}_2, \tilde{p}, \tilde{r}_{11}, \tilde{r}_{12}, \tilde{r}_{22}] = [W_1(x_2), iW_2(x_2), P(x_2), R_{11}(x_2), R_{12}(x_2), R_{22}(x_2)]\, e^{i\alpha(x_1-ct)}, \quad (20)$$

where α is the disturbance wavenumber (real), and c is the disturbance wave phase speed $= c_{\mathscr{R}}+ic_{\mathscr{I}}$ (complex). We are thus considering the temporal development of an instability at a particular x_1 location in the wake. The differential equations (with the prime denoting derivative with respect to x_2) for our unknowns become the following:

$$W_2' = -\alpha W_1, \quad (21)$$

$$i\alpha W_1(\bar{u}_1-c)+iW_2\bar{u}_1' = -i\alpha P-(i\alpha R_{11}+R_{12}'), \quad (22)$$

$$-\alpha W_2(\bar{u}_1-c) = -P'-(i\alpha R_{21}+R_{22}'), \quad (23)$$

Fourier decomposition of the stress (16)–(18) provides three simultaneous algebraic equations that can be solved for R_{11}, R_{12}, and R_{22} in terms of the velocity components W_1, W_2, W_1'. These results are of the form

$$R_{11} = \eta_1 W_1+\eta_2 W_2+\eta_3 W_1', \quad (24)$$

$$R_{12} = \eta_4 W_1+\eta_5 W_2+\eta_6 W_1', \quad (25)$$

$$R_{22} = \eta_7 W_1+\eta_8 W_2+\eta_9 W_1', \quad (26)$$

where the η's depend on the mean and turbulent quantities. The expressions for these coefficients are quite involved and not very illuminating, and will not be reproduced here. The important point is that substitution of (24)–(26) into (21)–(23) and the relation $dW_1/dx_2 = W'_1$ leads to a system of four simultaneous linear first-order differential equations for the unknown functions W_1, W'_1, W_2 and P.

For boundary conditions, far from the wake axis the disturbances must die out. Hence W_1, W'_1, W_2, $P \to 0$ as $x_2 \to 0$. At the wake axis the condition will depend on whether we consider symmetrical or asymmetrical disturbances. Our calculations have been performed for the asymmetrical case ($W'_2 = 0$ at $x_2 = 0$), since that is known to be the most unstable under laminar conditions (cf. Betchov & Criminale 1965). We also have $P = 0$, $W_1 = 0$ on the axis.

The dynamical equations together with the boundary conditions constitute a linear homogeneous system with the only possible nontrivial solution an eigensolution. We shall focus on the eigenvalues for neutral stability ($c_{\mathscr{I}} = 0$) for the turbulence-modified Orr–Sommerfeld problem. In a typical laminar stability calculation, the neutral curve is plotted in α, R (Reynolds number) co-ordinates, with $c_{\mathscr{R}}$ varying along the curve. We have neglected purely viscous effects and hence R is not a parameter in our calculations. Two parameters take the place of the Reynolds number in the present turbulent stability analysis, involving the intensity (q^2) and scale (Λ) of the background turbulence. There are several possible non-dimensional combinations, to be discussed in more detail in the following section. The two parameters in terms of which our stability calculations will be presented are $R_\epsilon = \Delta u_0 r_w/(q\Lambda)$ and Λ/r_w. Since the eddy viscosity tends to be proportional to $q\Lambda$, R_ϵ is an effective Reynolds number for the background turbulence. The ratio Λ/r_w obviously measures the fineness of the turbulence.

The mean velocity was taken to be a Gaussian for our calculations:†

$$\bar{u}_1 = 1 - \Delta \exp(-b_1 x_2^2/r_w^2), \tag{27}$$

where $b_1 = 0{\cdot}693$ and Δ is the axis velocity defect normalized by the free-stream velocity. The three normal components of the background turbulence were assumed equal $\overline{u'_1 u'_1} = \overline{u'_2 u'_2} = \overline{u'_3 u'_3} = \frac{2}{3}q^2$ and uniform across the wake. We used an eddy-viscosity approximation to specify the Reynolds stress of the background turbulence

$$\overline{u'_1 u'_2} = -q\Lambda \frac{\partial \bar{u}_1}{\partial x_2}. \tag{28}$$

Finally, the length scale Λ was taken to be constant across the wake.

The differential equations presented above were integrated using the rather standard fourth-order Runge–Kutta scheme with complex arithmetic. Gram–Schmidt orthogonalization was employed in order to preserve the linear independence of modes (see, for example, Betchov & Criminale 1965). This procedure was required for the eddy Reynolds numbers considered here, values up to $R_\epsilon = 2000$. Another means that we used for improving the numerical calculations of the Orr–Sommerfeld equation was to introduce a stretched independent variable η, defined by

$$\eta = x_2(\alpha|R_\epsilon|)^{\frac{1}{2}}. \tag{29}$$

This variable helps to normalize the various terms of the differential equations so that small errors will not be artificially amplified by the factor αR_ϵ appearing in the untransformed system (see, for example, Radbill & McCue 1970). Most of our results were

† Note that since the mean velocity profile for a wake resembles the mean velocity profile of a jet when one applies the appropriate translation of the frame of reference, we are also providing approximate jet solutions. A word of caution is appropriate however since the turbulence characteristics of jets are more non-homogeneous than the characteristics of wakes.

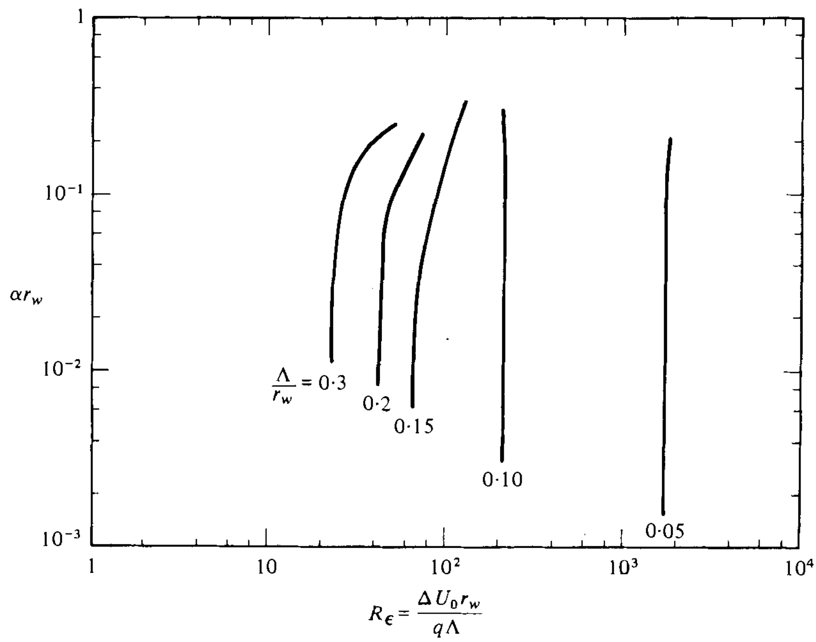

FIGURE 1. Computed neutral stability curves for a two-dimensional wake dominated by fine-scale turbulence.

generated using the CDC 7600 machine. A typical eigenvalue on a neutral curve required four iterations to converge and consumed approximately 8 seconds of computer time.

Figure 1 shows the computed results on the α, R_ϵ (wavenumber/eddy-viscosity based defect Reynolds number) plane with the turbulence scale size Λ as the parameter locating individual neutral curves ($c_{\mathscr{I}} = 0$). Note that the minimum critical eddy Reynolds number increases monotonically with the reduction of Λ. Since the stable regions in the α–R_ϵ plane are to the left of the neutral curves, these results imply that the two-dimensional incompressible turbulent wake dominated by fine scales becomes increasingly stable to imposed disturbances as the (fine) turbulence scale decreases. It should be noted here that the calculations have not been extended to larger values of the disturbance wavenumber α (or smaller values of the disturbance wavelength) due to the extreme numerical difficulty in obtaining such eigenvalues. However, it should be recalled that the above derivation assumed that the disturbance wavelengths are much larger than typical background turbulence length scales, so that the model becomes invalid at large wavenumbers (when $\alpha\Lambda$ is of order unity). On the contrary, it becomes rather easy to compute eigenvalues for small wavenumbers. Another point of note is that the calculations become very difficult for smaller values of Λ (or larger values of R_ϵ) than presented. In general, the product αR_ϵ controls the numerical aspects of the computation (see (29)); large values of α and R_ϵ provide the most difficulty.

The computed neutral stability curves (figure 1) become virtually independent of the wavenumber as the turbulence scale size relative to the wake radius becomes small. This is particularly evident for the $\Lambda/r_w = 0{\cdot}05$ neutral curve. On the other hand, as

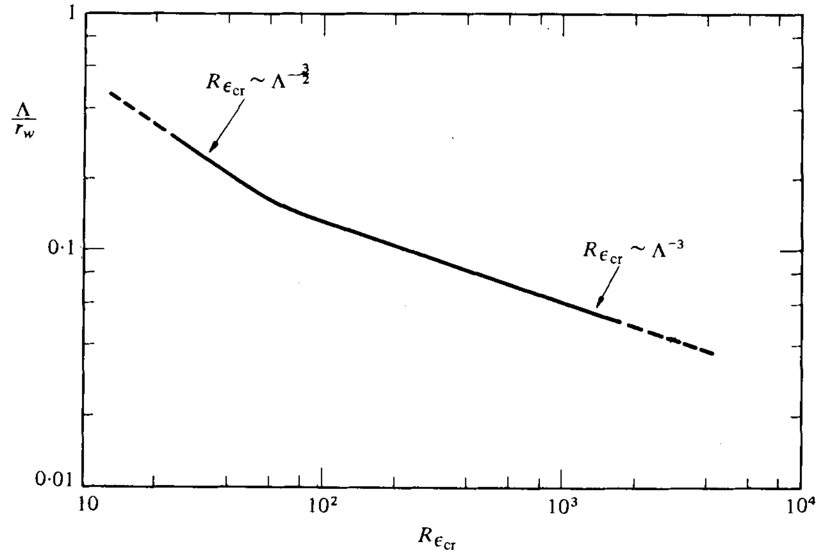

FIGURE 2. Variation of the critical eddy-viscosity based Reynolds number $R_{\epsilon_{cr}}$ with turbulence integral scale; $c_{\mathscr{R}} = -40{\cdot}0$.

Λ	α	$R_{\epsilon_{cr}}$	ϵ_{cr}	$q^2_{cr} = (\epsilon_{cr}/\Lambda)^2$
0·30	$1{\cdot}147 \times 10^{-2}$	23·0	$4{\cdot}35 \times 10^{-2}$	$2{\cdot}10 \times 10^{-2}$
0·25	1·004	29·8	3·35	1·80
0·23664	0·975	32·3	3·10	1·71
0·20	0·852	41·6	2·41	1·45
0·15	0·635	66·5	1·50	1·01
0·12896	0·447	100·0	1·00	0·601
0·10	0·301	215·0	0·465	0·216
0·09	0·272	288·0	0·348	0·149
0·08	0·246	406·0	0·246	0·0949
0·07	0·215	619·0	0·162	0·0533
0·06	0·182	1010·0	0·0990	0·0272
0·05	0·153	1720·0	0·0581	0·0135

TABLE 1. Critical Reynolds numbers ($c_{\mathscr{R}} = -40{\cdot}0$).

Λ/r_w increases, the wavenumber dependence becomes significant and the neutral curve tends to resemble the classical wake stability curve (cf. Reynolds 1972). However, the classical neutral curve does not envelop the curves calculated in the present paper, due to the more general nature of the stress–strain relationship in the presence of background turbulence. If we use the value of the Reynolds number in the range where the stability curve is insensitive to wavenumber, a cross-plot of the critical Reynolds number for each neutral curve versus the turbulence scale size effectively provides all the information of figure 1. In figure 2 we present this cross-plot and the eigenvalues which lead to this curve are given in table 1. Although we can offer no explanation, the stability calculations seem to indicate quite clearly that $R_{\epsilon_{cr}} \sim \Lambda^{-3}$ for smaller values of Λ.

4. On the behaviour of background turbulence

It is appropriate at this point to discuss the nature of the interaction between a wave perturbation and background turbulence, to the extent that it can be deduced from the model equations and computations presented above. First of all, we may compare the various terms on right-hand side of (13), for the wave-induced Reynolds stress, with the convection terms on the left side of that equation. In so doing, we use the boundary layer approximation for derivatives in the normal direction,

$$\partial/\partial x_2 \sim r_w^{-1},$$

and from (20) for streamwise derivatives of wave perturbation quantities, $\partial/\partial x_1 \sim \alpha$. We find that

$$\frac{\text{mean shear production}}{\text{convection}} = R_1 \sim \frac{\Delta\bar{u}_0}{\bar{u}}\frac{1}{\alpha r_w}, \tag{30a}$$

$$\frac{\partial\tilde{u}/\partial x_k \text{ terms}}{\text{convection}} = R_2 \sim \frac{q^2\tilde{U}}{\bar{u}\tilde{r}_{ij}}\frac{1}{\alpha r_w} \quad \text{or} \quad \frac{\overline{u_1'u_2'}\,\tilde{U}}{\bar{u}\tilde{r}_{ij}}\frac{1}{\alpha r_w}, \tag{30b}$$

$$\frac{\text{dissipation or } p' \text{ terms}}{\text{convection}} = R_3 \sim \frac{q/\bar{u}}{\Lambda/D_w}\frac{1}{\alpha r_w}. \tag{30c}$$

The term $\tilde{u}_k\,\partial\overline{u_i'u_j'}/\partial x_k$, representing inhomogeneity of the background turbulence, was not considered here although it is of the same order as the ratio expressed by (30*b*). As indicated by the calculations presented in figure 1, we are interested in wavenumbers satisfying $\alpha r_w < 1$. According to (30*a*) production by the mean shear will be small only if the mean shear is very small, $\Delta u_0/u \ll 1$. In (30*b*) there is no *a priori* reason to presume $q^2\tilde{U}/(\bar{u}\tilde{r}_{ij})$ or $\overline{u_1'u_2'}\,\tilde{U}/(\bar{u}\tilde{r}_{ij})$ to be small. And, with respect to (30*c*) if we envision background turbulence as being characterized by both small intensity and scale size, the dissipation and tendency-toward-isotropy terms are not easily neglected. Thus all of the terms in (13) should in general be retained, as was done in the calculations presented above.

Elastic limit

If the mean shear ($\Delta\bar{u}$) and the turbulent relaxation rate q/Λ are very small (as well as the Reynolds stress gradient $\partial\overline{u_i'u_k'}/\partial x_k$), that is $R_1 \ll 1$, $R_3 \ll 1$, (13) reduces to

$$\frac{D\tilde{r}_{ij}}{Dt} = -\left(\overline{u_j'u_k'}\frac{\partial\tilde{u}_i}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial\tilde{u}_j}{\partial x_k}\right) + (\bar{a}_{lj}^{mi} + \bar{a}_{li}^{mj})\frac{\partial\tilde{u}_l}{\partial x_m}. \tag{31}$$

This is the elastic limit, with the rate of generation of stress $(-\tilde{r}_{ij})$ proportional to the rate of strain $\partial\tilde{u}_j/\partial x_k$ and the effective modulus of elasticity involving the background turbulence intensity. If the turbulence is isotropic, then $\bar{s}_{ij} = 0$ and (11*b*) yields

$$\bar{a}_{lj}^{mi}\frac{\partial\tilde{u}_l}{\partial x_m} = \tfrac{6}{15}q^2\frac{\partial\tilde{u}_i}{\partial x_j} \tag{32}$$

and (31) becomes

$$\frac{D\tilde{r}_{ij}}{Dt} = -\tfrac{4}{15}q^2\left(\frac{\partial\tilde{u}_i}{\partial x_j} + \frac{\partial\tilde{u}_j}{\partial x_i}\right). \tag{33}$$

This is precisely the result of Crow (1968). The result in this limit is exact in the sense that no closure is required; Crow (1968) also derived (32) from the requirements of continuity and isotropy. It should be noted that the conditions under which this limit applies are quite restrictive.

Viscous limit

Here we again require homogeneous mean ($\partial\overline{u}/\partial x_2 = 0$) and turbulent ($\partial\overline{u_1' u_2'}/\partial x_2 = 0$) flows. Also the wavenumber α must be low so that the convective term is small. To be precise, the viscous limit requires $R_2 \gg 1$, $R_3 \gg 1$, in which case (13) reduces to

$$0 = -\left(\overline{u_j' u_k'}\frac{\partial\tilde{u}_i}{\partial x_k} + \overline{u_i' u_k'}\frac{\partial\tilde{u}_j}{\partial x_k}\right) + (\overline{a}_{lj}^{mi} + \overline{a}_{li}^{mj})\frac{\partial\tilde{u}_l}{\partial x_m}$$
$$-k_p\frac{q}{\Lambda}(\tilde{r}_{ij} - \tfrac{1}{3}\tilde{r}_{kk}\delta_{ij}) - \tfrac{1}{6}k_d\frac{q}{\Lambda}\tilde{r}_{kk}\delta_{ij}. \quad (34)$$

If we again assume isotropy for the background turbulence, (34) yields for $\tilde{r}_{12}$

$$\tilde{r}_{12} = -\frac{4}{15}\frac{1}{k_p}q\Lambda\left(\frac{\partial\tilde{u}_1}{\partial x_2} + \frac{\partial\tilde{u}_2}{\partial x_1}\right). \quad (35)$$

This is clearly of the viscous form with an effective viscosity of $(\frac{4}{15}k_p)\,q\Lambda$. Such a value is quite close to the eddy viscosity; many authors, for example Ng & Spalding (1972), have used the approximation $\epsilon \sim q\Lambda$, and the coefficient $\frac{4}{15}k_p^{-1}$ is of order unity.

Equation (35) shows that the viscous approximation used by Hussain & Reynolds (1972) and Reynolds (1972) can be derived from the governing relation for $\tilde{r}_{ij}$. However, as with the elastic limit, this limit can be justified only with some rather drastic assumptions. At least in the absence of mean shear, the general behaviour is viscoelastic as recognized by Crow (1968). He also discussed the fact that the turbulence relaxation rate q/Λ determines whether the behaviour is elastic or viscous, but the viscous aspect cannot be addressed without introducing closure for the relaxation terms, as is done here.

General viscoelastic behaviour

One simple type of non-Newtonian fluid is the Maxwellian fluid, in which the stress (τ_{ij}) obeys the following constitutive relation:

$$T_1\frac{\partial\tau_{ij}}{\partial t} = \rho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \tau_{ij}, \quad (36)$$

where T_1 is a time constant. To determine the effective (complex) viscosity, assume that the velocity and stress are proportional to $\exp i\alpha(x-ct)$, as in (20). It is then a straightforward matter (see, for example, Betchov & Criminale 1965) to show that the complex viscosity $\nu_0 e^{-i\theta}$ is given by

$$\nu_0 = \nu/(1+(\alpha c T_1)^2)^{\frac{1}{2}}, \quad \theta = -\tan^{-1}(\alpha c T_1). \quad (37)$$

For the fine-scaled turbulent fluid considered here, the constitutive relation follows from (17) above. If the terms representing mean shear and inhomogeneity of the background turbulence are omitted, and if we set $\tau_{ij} = -\tilde{r}_{ij}$, then (17) becomes

$$\frac{\partial\tau_{ij}}{\partial t} + \overline{u}_1\frac{\partial\tau_{ij}}{\partial x_1} = k_p\frac{q}{\Lambda}\left\{\frac{4}{15}\frac{q\Lambda}{k_p}\left(\frac{\partial\tilde{u}_i}{\partial x_j} + \frac{\partial\tilde{u}_j}{\partial x_i}\right) - \tau_{ij}\right\}. \quad (38)$$

This form is similar to that of the Maxwellian fluid, and the modulus and argument of the viscosity are found to be

$$\nu_0 = \frac{\frac{4}{15}\frac{q\Lambda}{k_p}}{\left[1+\frac{1}{k_p^2}\frac{\Lambda^2}{q^2}\alpha^2(\bar{u}_1-c)^2\right]^{\frac{1}{2}}}, \tag{39a}$$

$$\theta = -\tan^{-1}\left\{-\frac{\alpha(\bar{u}_1-c)}{kq/\Lambda}\right\}. \tag{39b}$$

In our calculations $\bar{u}_1 - c = \bar{u}_1 - c_{\mathscr{R}}$ since only the neutral curves ($c_{\mathscr{I}} \equiv 0$) are obtained. Furthermore, $\bar{u}_1 - c_{\mathscr{R}}$ was found to vary between $1{\cdot}10\Delta$ and 40Δ, and as a result the phase angles are positive. Positive phase angles are known to be stabilizing, which helps to explain the neutral stability calculations.

Role of mean shear

Unfortunately, this viscoelastic description seriously oversimplifies the nature of background turbulence, because mean shear effects tend to be quite important. To illustrate this, we considered solving (13) for $\tilde{r}_{ij}$ with the wave velocity $\tilde{u}$ *imposed*, of the form given by (20). This is course is not proper since $\tilde{u}$ and $\tilde{r}_{ij}$ must be solved for simultaneously. That was the subject of the previous section, so this exercise will not be dwelt upon except to point out one important result.

To keep matters simple we assumed $\tilde{r}_{11} = \tilde{r}_{22} = \tilde{r}_{33}$, so that (13) results in two coupled equations for $\tilde{r}_{11}$ and $\tilde{r}_{12}$. It was further assumed that the background turbulence is isotropic. The two equations are linear in $\tilde{r}_{11}$, $\tilde{r}_{12}$, and the solution consists of homogeneous and particular terms. The latter are of the form (20), with R_{11} and R_{12} expressed in terms of rather intricate algebraic expressions that are not particularly enlightening. The homogeneous terms are of the form

$$\tilde{r}_{ij,h} = f(x-\bar{u}_1 t)\, e^{\alpha' x}, \tag{40}$$

where α' is given by

$$\bar{u}_1^2\alpha'^2 + (k_p + \tfrac{3}{2}k_d)\frac{q}{\Lambda}\bar{u}_1\alpha' + \tfrac{3}{2}k_p k_d \frac{q^2}{\Lambda^2} - \frac{4}{15}\left(\frac{\partial \bar{u}_1}{\partial x_2}\right) = 0. \tag{41}$$

The roots of this equation are real, and there is a positive root if

$$\frac{\partial \bar{u}_1}{\partial x_2} > (\tfrac{45}{8}k_p k_d)^{\frac{1}{2}}\frac{q}{\Lambda} \simeq \frac{q}{\Lambda}. \tag{42}$$

Thus there is the possibility of an exponentially growing solution if the mean shear rate is greater than the relaxation rate of the turbulence. This inequality may or may not be satisfied in realistic situations. It should be noted that there are practical upper limits to the value of q/Λ, since very intense and very fine-scaled turbulence would tend to dissipate rapidly. This does not necessarily mean that the wave stress would in reality grow indefinitely, for we have assumed the wave velocity to have a constant amplitude in this exercise. It does, however, indicate that the mean shear may be expected to have an important effect if $\partial\bar{u}_1/\partial x_2 \gtrsim q/\Lambda$, and that the general behaviour is not simply viscoelastic.

5. Concluding remarks

A theoretical treatment has been presented for the interaction between fine-scaled background turbulence and potential large-scale disturbances in shear flows. The second-order closure approximations that are introduced are not well substantiated, and the resulting equations should probably be considered as no more than model equations. However, the results do shed light on the nature of the interaction between large-scale disturbances and background turbulence. In the absence of mean shear and turbulence inhomogeneities, the general behaviour is viscoelastic, as suggested by earlier workers. The behaviour is more complicated if mean shear or inhomogeneities exist, in which case second-order closure is essential. The linear stability calculations that are reported for an idealized wake flow contain the interesting result that stabilization increases as the turbulent scale size becomes finer (see figure 1).

Bradshaw (1966) observed that a similar process of stabilization occurs in the near field of plane mixing layers. If the boundary layer on the splitter plate is laminar, transition is observed just beyond the separation point and a fully turbulent mixing layer is quickly established. But if the boundary layer is turbulent, only a slow relaxation to the fully turbulent mixing layer occurs, requiring much larger distances than if the shear layer originates from a laminar boundary layer. Several recent studies have documented the nature of this dependence on initial conditions for both plane and axisymmetric mixing layers (Hussain & Clark 1977; Hussain & Zedan 1978*a*, *b*; Husain & Hussain 1979). One-dimensional frequency spectra obtained by Husain & Hussain (1979) clearly show the development of instabilities, harmonics, etc., in the laminar case. The initially turbulent case, on the other hand, shows a monotonic evolution of the spectrum with no evidence of any discrete modes. Recent flow visualization studies of high Reynolds number mixing layers (Clark & Hussain 1979) have revealed that the organization of the coherent structure in a turbulent mixing layer is quite stable when it is initially fully turbulent, compared to cases when the layer is initially laminar. This last study seems to illustrate what our model has demonstrated; however, to our knowledge, there is no direct experimental support for the model presented here. A careful incompressible experiment to determine wake stability boundaries over a range of Λ/r_w is clearly desirable. Whether generated by a grid or by the boundary layer on the wake-producing body, the resulting scale sizes should be 3–30 times smaller than normal wake turbulent scale sizes. The Reynolds number of the flow would have to be quite high so that the background turbulence would not decay by viscous dissipation over wake distances of interest. A brief attempt was made by J. A. Woodroffe, P. I. Singh and H. H. Legner (1974, private communication) to stabilize a jet by introducing grid turbulence, with inconclusive results, most likely due to the marginal Reynolds number of the flow.

A realistic attempt at examining the features of a supersonic wake under conditions where the wake is apparently stabilized was undertaken by Avidor & Schneiderman (1975). This experiment involved the axisymmetric wake produced by a sting in a Ludwieg tube with a Mach number of 2·5; the sting boundary layer was fully turbulent, with a thickness comparable to the sting diameter. At the lowest Reynolds numbers (based upon sting diameter), $Re_D \sim 7{\cdot}5\times 10^4$, they observed the classical $x^{\frac{1}{3}}$ wake growth from the near wake. As the Reynolds number was increased, longer portions of non-growing (stable in the present context) wake were seen, up to a distance of 50 diameters at the highest Reynolds numbers tested, $Re_D = 1{\cdot}5\times 10^6$. Detailed laser velocimetry was performed at an intermediate condition, $Re_D = 6\times 10^5$, where the wake was apparently stabilized for 25 diameters. From these measurements, the near wake scale size (macroscale) was estimated to be about $\frac{1}{3}$ of the self-similar wake value, and the r.m.s. fluctuating velocity was shown to be perhaps 50 %. At the lowest Reynolds number, where no wake stabilization was evident, the near wake scale size was approximately equal to the self-similar value. The trend observed in the Avidor & Schneiderman experiment is thus qualitatively consistent with the present stability calculations. However, no detailed comparisons can be made because only incompressible, two-dimensional wakes have been modelled and the experimentally observed scale sizes are larger than the values considered in figures 1 and 2. It is interesting to note that conditions for the hypersonic cone near wakes considered by Finson (1973) correspond to $q/\Delta u_0 \simeq 0{\cdot}04$, $\Lambda/r_w \simeq 0{\cdot}03$, $R_\epsilon \simeq 800$. While it is extremely unfair to compare these values with the results given here, they indeed fall in the stable regions of figures 1 and 2.

In addition to the hypersonic wake behaviour that suggested this effort, other interesting observations implying a stabilizing effect due to background turbulence should be noted. Reduction of jet noise through turbulence suppression is one. It has been well substantiated that the breaking up, or suppression, of the largest eddies will reduce far-field noise levels (see, for example, Berenak 1960). This has been achieved by altering the jet exhaust nozzle into several smaller nozzles. In relation to the present model, the smaller nozzles establish smaller turbulence scales that effectively suppress the generation of large-scale turbulence (instabilities). There may also be application of the concepts discussed here to the dynamics of the large-scale structures that have been studied by Brown, Roshko, and co-workers (e.g. Brown & Roshko 1974), since these structures are generally superimposed on a smaller-scaled background turbulence field at high Reynolds numbers.

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